On certain families of naturally graded Lie algebras[*]

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Abstract

In this work large families of naturally graded nilpotent Lie algebras in arbitrary dimension and characteristic sequence $(n, q, 1)$ with $n \equiv 1 \,(mod\,2)$ satisfying the centralizer property are given. This centralizer property constitutes a generalization, for any nilpotent algebra, of the structural properties characterizing the Lie algebra Q_n. By considering certain cohomological classes of the space $H^2(\mathfrak{g}, \mathbb{C})$, it is shown that, with few exceptions, the isomorphism classes of these algebras are given by central extensions of Q_n by $\mathbb{C}^p$ which preserve the nilindex and the natural graduation.
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Introduction

The first systematic results about naturally graded nilpotent Lie were obtained in the sixties in the context of the analysis of the variety of nilpotent Lie algebra laws. In 1966 Vergne [9] concentrated on the less nilpotent Lie algebras, which she called filiform, and classified the naturally graded Lie algebras having this property. This gave the key to a first estimation of the number of irreducible components of the variety [10]. The classification result was, in a certain manner, surprising : there are only two models, called respectively L_n and Q_n, where the second exists only in even dimension. The filiform model Lie algebra, as L_n is called usually, is without doubt the most and best studied nilpotent Lie algebra over the last thirty years. Most studies dedicated to filiform Lie algebras and its deformations are dedicated to this model. For this reason, the

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algebra Q_n has often been relegated to a secondary position. However, the nonexistence of this algebra in odd dimensions makes it, from a structural point of view, much more interesting than the model algebra L_n. In fact, when considering the centralizers $C_{Q_n} I$ of ideals I of the central descending sequence in Q_n, we find an intriguing property, namely, that the integer part of the nilindex modulus 2 works as frontier between those ideals of the sequence contained in its associated centralizer in Q_n and those whose intersection of the centralizer with the complementary of the ideal in the algebra is nonzero. This fact can be used to estimate how far is Q_n from having an abelian commutator algebra; the index above tells that this algebra is as far as posiible from having it. This question leads naturally to search for a generalization of this property for other naturally graded nilpotent Lie algebras. Making use of the characteristic sequence of an algebra, we can concentrate on concrete classes of algebras. The cohomology space $H^2(\mathfrak{g}, \mathbb{C})$ of $\mathfrak{g}$ with values in the base field are of wide interest to determine particular classes of central extensions of $\mathfrak{g}$ by $\mathbb{C}$ which preserve either the natural graduation or any other property of the extended algebra. By introducing a partition of this space, we are in situation of isolating the cohomology classes which make direct reference to the property of the centralizers. This can be used to achieve a complete classification, for a fixed characteristic sequence, of algebras bahaving as Q_n does. This is done for the sequences $(2m-1, 1, 1)$ and $(2m-1, 2, 1)$ in arbitrary dimension, and can be applied to any other sequence. Our main purpose is, however, to provide families, in arbitrary dimensions and characteristic sequences $(n, q, 1)$ with $n \equiv 1 \, (mod \, 2)$ and $q \geq 1$, of naturally graded nilpotent Lie algebras with this centralizer property. The obtained algebras can be interpreted as the analogue, for its corresponding characteristic sequence, of the Lie algebra Q_n.

1   Preliminaries and notations

Whenever we speak about Lie algebras in this work, we refer to finite dimensional complex Lie algebras.

Definition. *Let $\mathfrak{g}$ be a finite dimensional vectorial space over $\mathbb{C}$. A Lie algebra law over $\mathbb{C}^n$ is a bilinear alternated mapping $\mu \in Hom(\mathbb{C}^n \times \mathbb{C}^n, \mathbb{C}^n)$ which satisfies the conditions*

  1. *$\mu(X, X) = 0, \ \forall \, X \in \mathbb{C}^n$*

  2. *$\mu(X, \mu(Y, Z)) + \mu(Z, \mu(X, Y)) + \mu(Y, \mu(Z, X)) = 0, \ \forall \, X, Y, Z \in \mathbb{C}^n$, (Jacobi identity)*

If μ is a Lie algebra law, the pair $\mathfrak{g} = (\mathbb{C}^n, \mu)$ is called Lie algebra. From now on we identify the Lie algebra with its law μ.

Remark 1. *We say that μ is the law of $\mathfrak{g}$, and where necessary we use the bracket notation to describe the law :*

$$[X, Y] = \mu(X, Y), \ \forall \ X, Y \in \mathfrak{g}$$

The nondefined brackets are zero or obtained by antisymmetry.

To any Lie algebra we can associate the following sequence :

$$C^0\mathfrak{g} = \mathfrak{g} \supset C^1\mathfrak{g} =\supset C^2\mathfrak{g} = \left[C^1\mathfrak{g}, \mathfrak{g}\right] \supset ... \supset C^k\mathfrak{g} = \left[C^{k-1}\mathfrak{g}, \mathfrak{g}\right] \supset ...$$

called the descending central sequence of $\mathfrak{g}$.

Definition. *A Lie algebra $\mathfrak{g}$ is called nilpotent if there exists an integer (called nilindex $n(\mathfrak{g})$ of $\mathfrak{g}$) $k \geq 1$ such that $C^k\mathfrak{g} = \{0\}$ and $C^{k-1}\mathfrak{g} = \{0\}$.*

Definition. *An n-dimensional nilpotent Lie algebra is called filiform if*

$$\dim C^k\mathfrak{g} = n - k - 1, \ 1 \leq k \leq n - 1$$

Remark 2. *Calling $p_i = dim \left(\frac{C^{i-1}\mathfrak{g}}{C^i\mathfrak{g}}\right)$ for $1 \leq \leq n(\mathfrak{g})$, the type of the nilpotent Lie algebra is the sequence $\{p_1, .., p_r\}$. Then a filiform algebra corresponds to those of type $\{2, 1, .., 1\}$ [10].*

We recall the laws for the $(n+1)$-dimensional filiform Lie algebras L_n and Q_n, which are the only ones we will use here :

1. $L_n \ (n \geq 3)$:

$$[X_1, X_i] = X_{i+1}, \ 2 \leq i \leq n$$

over the basis $\{X_1, .., X_{n+1}\}$.

2. $Q_{2m-1} \ (m \geq 3)$:

$$[X_1, X_i] = X_{i+1}, \ 1 \leq i \leq 2m - 1$$
$$[X_j, X_{2m+1-j}] = (-1)^j X_{2m}, \ 2 \leq j \leq m$$

over the basis $\{X_1, .., X_{2m}\}$.

Definition. *A Lie algebra $\mathfrak{g}$ is graded over $\mathbb{Z}$ if it admits a decomposition*

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$$

where the $\mathfrak{g}_k$ are $\mathbb{C}$-subspaces of $\mathfrak{g}$ which satisfy $[\mathfrak{g}_r, \mathfrak{g}_s] \subset \mathfrak{g}_{r+s}$, $r, s \in \mathbb{Z}$.

Observe that any graduation defines a sequence

$$S_k = F_k(\mathfrak{g}) = \bigoplus_{t \geq k} \mathfrak{g}_t$$

with the properties

1. $\mathfrak{g} = \bigsqcup S_k$

2. $[S_i, S_j] \subset S_{i+j} \ \forall i, j$

3. $S_i \subset S_j \ $ si $i > j$

Definition. *A family $\{S_i\}$ of subspaces of $\mathfrak{g}$ define a filtration (descending) over $\mathfrak{g}$ if it satisfies properties $1), 2), 3)$. The algebra is called filtered.*

The construction can be reversed, i.e., any filtration defines a graduation by taking $\mathfrak{g}_k = \frac{S_{k-1}}{S_k}$ for $k \geq 1$. The graduation is called associated to the filtration $\{S_i\}$ and it defines a Lie algebra with the rule

$$\left[\bar{X}_i, \bar{X}_j \right] = \overline{[X_i, X_j]},$$

where $\bar{X}_k = X_k \pmod{S_{k+1}}$ for $k = i, j$ and $\overline{[X_i, X_j]} = [X_i, X_j] \pmod{S_{i+j+1}}$

Definition. *A nilpotent Lie algebra is called naturally graded if $\mathfrak{g} \simeq \mathfrak{gr}(\mathfrak{g})$, where $\mathfrak{gr}(\mathfrak{g})$ is the graduation associated to the filtration induced in $\mathfrak{g}$ by the central descending sequence.*

It follows immediately that both L_n and Q_n are naturally graded. They are in fact the only filiform Lie algebras having this property [10].

Let $\mathfrak{g}_n = (\mathbb{C}^n, \mu)$ be a nilpotent Lie algebra. For any nonzero vector $X \in \mathfrak{g}_n - C^1\mathfrak{g}_n$ let $c(X)$ be the ordered sequence of a similitude invariant for the nilpotent operator $ad_\mu(X)$; i.e., the ordered sequence of dimensions of Jordan blocks of this operator. The set of these sequences is ordered lexicographically.

Definition. *The characteristic sequence of $\mathfrak{g}_n$ is an isomorphism invariant $c(\mathfrak{g}_n)$ defined by*

$$c(\mathfrak{g}_n) = \max_{X \in \mathfrak{g}_n - C^1\mathfrak{g}_n} \{c(X)\}$$

A nonzero vector $X \in \mathfrak{g}_n - C^1\mathfrak{g}_n$ for which $c(X) = c(\mathfrak{g}_n)$ is called characteristic vector.

Remark 3. *In particular, the algebras with maximal characteristic sequence $(n-1, 1)$ correspond to the filiform algebras introduced by Vergne.*

It is often convenient to use the so called ontragradient representation of a Lie algebra $\mathfrak{g}$. Let $n = dim(\mathfrak{g})$ and $\{X_1, .., X_n\}$ be a basis. If $C^k_{i,j}$ are the structure constants of the algebra law μ, we can define, over the dual basis $\{\omega_1, .., \omega_n\}$, the differential

$$d_\mu \omega_i (X_j, X_k) = -C^i_{jk}$$

Then the Lie algebra is rewritten as

$$d\omega_i = -C^i_{jk}\omega_j \wedge \omega_k \ 1 \le i, j, k \le n,$$

The Jacobi condition is equivalent to $d^2\omega_i = 0$ for all i.

Remark 4. *In what follows we will use the preceding form, up to the sign, to describe Lie algebras. It will be seen that certain structural properties are better seen by using this form. For example, this was the procedure to analyze the Lie algebra models [4].*

1.1 The spaces $H^2(\mathfrak{g}, \mathbb{C})$

Recall that the space $H^2(\mathfrak{g}, \mathbb{C}^p)$ can be interpreted as the space of classes of p-dimensional central extensions of the Lie algebra $\mathfrak{g}$. We recall the elementary facts :

Let $\mathfrak{g}$ be an n-dimensional nilpotent Lie algebra with law μ_0. A central extension of $\mathfrak{g}$ by $\mathbb{C}^p$ is an exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C}^p \longrightarrow \bar{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

such that $\mathbb{C}^p \subset Z(\bar{\mathfrak{g}})$. Let α be a cocycle of the De Rham cohomology $Z^2(\mathfrak{g}, \mathbb{C}^p)$. This gives the extension

$$0 \longrightarrow \mathbb{C}^p \longrightarrow \mathbb{C}^p \oplus \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0$$

with associated law $\mu = \mu_0 + \alpha$ defined by

$$\mu((a, x), (b, y)) = (\alpha\mu_0(x, y), \mu_0(x, y))$$

In the following we are only interested in extensions of $\mathbb{C}$ by $\mathfrak{g}$, i.e, extensions of degree one. It is well known that the space of 2-cocycles $Z^2(\mathfrak{g}, \mathbb{C})$ is identified with the space of linear forms over $\bigwedge^2 \mathfrak{g}$ which are zero over the subspace Ω :

$$\Omega := \langle \mu_0(x, y) \wedge z + \mu_0(y, z) \wedge x + \mu_0(z, x) \wedge y \rangle_{\mathbb{C}}$$

The extension classes are defined modulus the coboundaries $B^2(\mathfrak{g}, \mathbb{C})$. This allows to identify the cohomology space $H^2(\mathfrak{g}, \mathbb{C})$ with the dual of the space $\frac{Ker\,\lambda}{\Omega}$, where $\lambda \in Hom\left(\bigwedge^2 \mathfrak{g}, \mathfrak{g}\right)$ is defined as

$$\lambda(x \wedge y) = \mu_0(x, y) \ x, y \in \mathfrak{g}$$

In fact we have $H_2(\mathfrak{g}, \mathbb{C}) = \frac{Ker\lambda}{\Omega}$ for the 2-homology space, and as $H^2(\mathfrak{g}, \mathbb{C}) = Hom_{\mathbb{C}}(H_2(\mathfrak{g}, \mathbb{C}), \mathbb{C})$ the assertion follows.

Notation 1. *Let* $\varphi_{ij} \in H^2(\mathfrak{g}, \mathbb{C})$ *the cocycles defined by*

$$\varphi_{ij}(X_k, X_l) = \delta_{ik}\delta_{jl}$$

Observe that a cocycle φ can be written as a linear combination of the preceding cocycles. We have :

Lemma. $\sum a^{ij}\varphi_{ij} = 0$ *if and only if* $\sum a^{ij}(X_i \wedge X_j) \in \Omega$

Let $\mathfrak{g}$ be an n-dimensional nilpotent Lie algebra. The subspace of central extensions is noted by $E_{c,1}(\mathfrak{g})$. It has been shown that this space is irreducible and constructible. However, for our purpose this space is too general. We only need certain cohomology classes of this space.

Notation 2. *For* $k \geq 2$ *let*

$$H_k^{2,t}(\mathfrak{g}, \mathbb{C}) = \left\{ \varphi_{ij} \in H^2(\mathfrak{g}, \mathbb{C}) \mid i+j = 2t+1+k \right\}, \ 1 \leq t \leq \left[\frac{n-3}{2}\right],$$

$$H_k^{2,\frac{t}{2}}(\mathfrak{g}, \mathbb{C}) = \left\{ \varphi_{ij} \in H^2(\mathfrak{g}, \mathbb{C}) \mid i+j = t+1+k \right\}, \ t \in \left\{1, .., \left[\frac{n-3}{2}\right]\right\}, t \equiv 1 \, (\text{mod } 2)$$

These cocycles are essential to determine the central extensions which are additionally naturally graded. If $\mathbf{E}_{c,1}(\mathfrak{g})$ denotes the central extensions that are naturally graded, we consider the subspaces

$$\mathbf{E}_{c,1}^{t,k_1,...,k_r}(\mathfrak{g}) = \left\{ \mu \in \mathbf{E}_{c,1}(\mathfrak{g}) \mid \mu = \mu_0 + \left(\sum \varphi_{ij}^{k_i}\right), \ \varphi_{ij}^{k_i} \in H_{k_i}^{2,t}(\mathfrak{g}, \mathbb{C}) \right\}$$

$$\mathbf{E}_{c,1}^{\frac{t}{2},k_1,...,k_r}(\mathfrak{g}) = \left\{ \mu \in \mathbf{E}_{c,1}(\mathfrak{g}) \mid \mu = \mu_0 + \left(\sum \varphi_{ij}^{k_i}\right), \ \varphi_{ij}^{k_i} \in H_{k_i}^{2,\frac{t}{2}}(\mathfrak{g}, \mathbb{C}) \right\}$$

where $0 \leq k_j \in \mathbb{Z}$, $j = 1, .., r$.
Given a basis $\{X_1, .., X_n, X_{n+1}\}$ of μ belonging to any of these spaces, the Lie algebra law is defined by :

$$\mu(X_i, X_j) = \mu_0(X_i, X_j) + \left(\sum \varphi_{ij}^k\right) X_{n+1}, \ 1 \leq i,j \leq n$$

Lemma. *As vector spaces, the following identity holds :*

$$\mathbf{E}_{c,1}(\mathfrak{g}) = \sum_{t,k} \mathbf{E}_{c,1}^{t,k_1,...,k_r}(\mathfrak{g}) + \mathbf{E}_{c,1}^{\frac{t}{2},k_1,...,k_r}(\mathfrak{g})$$

The proof is elementary. Observe that, though t is bounded by the dimension, $k \geq 2$ has no restrictions. However, the sum is finite, for the spaces $\mathbf{E}_{c,1}^{t,k_1,..,k_r}$ are zero for almost any choice $(k_1, .., k_r)$.
Given the Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \mu_0)$, we have the associated graduation $\mathfrak{gr}(\mathfrak{g}) = \sum_{i=1}^{n(\mathfrak{g})} \mathfrak{g}_i$, where $\mathfrak{g}_i = \frac{C^{i-1}\mathfrak{g}}{C^i\mathfrak{g}}$ and $n(\mathfrak{g})$ is the nilindex of $\mathfrak{g}$. Independently of $\mathfrak{g}$ being naturally graded or not, any vector X has a fixed position in one of the graduation blocks.

Remark 5. *The study of the central extensions which preserve a graduation is reduced to the study of the position of the adjoined vector X_{n+1}. Note that in this sense the cocycles $\varphi_{ij} \in H_k^{2,t}(\mathfrak{g}, \mathbb{C})$ codify this information.*

2 The centralizer property for naturally graded Lie algebras

For the filiform Lie algebras, Vergne proved that there exist, up to isomorphism, only two clases of naturally graded Lie algebras, L_n and Q_n, for which the second only exists in even dimension. Now the model Q_n has an interesting structural property that explains its nonexistence in odd dimension : for $p \geq \left[\frac{n-1}{2}\right]$, where [ ] denotes the integer part function, the centralizer $C_{Q_n}(C^p Q_n)$ in Q_n contains the ideal $C^p Q_n$, while for $q \leq p$ we have $C_{Q_n}(C^p Q_n) \supsetneq C^p Q_n$. This property gives an estimation of how far the algebra Q_n is from having an abelian commutator algebra, as happens for L_n. This fact suggestts to study the Lie algebras $\mathfrak{g}$ which satisfy the following property :

$$
\begin{aligned}
C_{\mathfrak{g}}(C^p \mathfrak{g}) &\supseteq C^p \mathfrak{g}, \, p \geq \left[\frac{n(\mathfrak{g})}{2}\right] \\
C_{\mathfrak{g}}(C^p \mathfrak{g}) &\not\supseteq C^p \mathfrak{g}, \, p < \left[\frac{n(\mathfrak{g})}{2}\right]
\end{aligned}
\tag{1}
$$

where $n(\mathfrak{g})$ is the nilpotence class (or nilindex) of $\mathfrak{g}$. We will call (P) the

centralizer property.

Now, a detailed analysis shows that (P) can be satisfied in two manners :

1. There exist $X, Y \in C^{\left[\frac{n(\mathfrak{g})}{2}\right]-1}\mathfrak{g} - C^{\left[\frac{n(\mathfrak{g})}{2}\right]}\mathfrak{g}$ such that $[X, Y] \neq 0$,

2. if there are $X, Y \in C^{\left[\frac{n(\mathfrak{g})}{2}\right]-1}\mathfrak{g}$ with $[X, Y] \neq 0$, then either $X \in C^{\left[\frac{n(\mathfrak{g})}{2}\right]}\mathfrak{g}$ or $Y \in C^{\left[\frac{n(\mathfrak{g})}{2}\right]}\mathfrak{g}$.

We have also to distinguish between two classes of algebras :

Definition. *A naturally graded Lie algebra satisfying the centralizer property through condition 1. (thus not verifying 2.) is called a $(P1)$-algebra.*
A naturally graded Lie algebra satisfying the centralizer property through condition 2. (thus not verifying 1.) is called a $(P2)$-algebra.

Remark 6. *It seems that condition 1) is less natural than 2), for it implies a nonzero bracket in a specific graduation block of $\mathfrak{g}$. However, we will see that even those algebras satisfying 1) are obtained in a "natural" manner.*

2.1 $(P1)$-algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of finite dimension, $\mathfrak{h}$ a Cartan sub-algebra, Φ a root system associated to $\mathfrak{h}$ and Δ a basis of simple roots. We introduce a partial ordering in $\mathfrak{h}^*$ relative to which the elements are called positive if their are linear combinations of simple roots with nonnegative coefficients [8]. Thus, respect to this ordering, we have

$$\Phi = \Phi^+ \cup \Phi^-$$

Recall that the subalgebra $\mathfrak{h}$ induces the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} + \coprod_{\alpha \in \Phi} L_\alpha$$

into weight spaces.

Let $ht : \Phi \to \mathbb{Z}^+$ be the height function. We can define the sets

$$\Delta(k) = \left\{ \alpha \in \Phi^+ \mid ht(\alpha) = k \right\}$$

These sets are of importance, as they give us a natural graduation of the nil-radical of a standard Borel subalgebra $\mathfrak{b}(\Delta)$.

Theorem 1. *Let $\mathfrak{n}$ be the nilradical of a standard Borel subalgebra $\mathfrak{b}(\Delta)$ of a complex simple Lie algebra distinct from G_2. Then $\mathfrak{n}$ satisfies (P) and $1)$.*

The proof is an immediate consequence of the following result :

Proposition 1. *Let $\mathfrak{n}$ be the nilradical of a standard Borel subalgebra $\mathfrak{b}(\Delta)$ of a complex simple Lie algebra distinct from G_2. Let $p = ht(\delta)$ be the height of the maximal root. Then there exist roots α, β whose height is $[\frac{ht(\delta)}{2}]$ such that $\alpha + \beta$ is a positive root.*

Proof.   1. $\mathfrak{g} = A_l$:

    (a) $l = 2q$. Let δ be the maximal root. For $1 \leq t \leq q - 2$ we have

$$(\delta - \alpha_{2q} - \alpha_{2q-1} - .. - \alpha_{2q-t}, \alpha_{2q-t-1}) = 1$$

        which proves that $\omega_1 = \alpha_1 + .. + \alpha_q$ is a root. In the same way it is seen that $\omega_2 = \alpha_{q+1} + .. + \alpha_{2q}$ is also a root. We have $ht(\omega_1) = ht(\omega_2) = q$, thus $\omega_1, \omega_2 \in \Delta(q)$.

    (b) $l = 2q + 1$. Reasoning as before, it follows that

$$\omega_1 = \alpha_1 + .. + \alpha_q, \ \omega_2 = \alpha_{q+1} + .. + \alpha_{2q} \in \Delta(q)$$

        and $\omega_1 + \omega_2 = \delta - \alpha_n \in \Phi$, as α_n is a particular root.

2. $\mathfrak{g} = B_l$: For $1 \leq t \leq l - 2$ we have

$$(\delta - \alpha_2 - .. - \alpha_t, \alpha_{t+1}) > 0$$

so that $\omega_1' = \delta - (\alpha_2 + .. + \alpha_{l-1})$. Now $(\omega_1', \alpha_1) > 0$ and $(\omega_1' - \alpha_1, \alpha_2) > 0$, thus

$$\omega_1 = \alpha_3 + .. + \alpha_{l-1} + 2\alpha_l \in \Delta\,(l - 1) = \Delta\left(\left[\frac{ht\,(\delta)}{2}\right]\right)$$

Considering the α_t-string through $(\alpha_1 + .. + \alpha_{t-1})$ $(2 \leq t \leq l - 1)$ we obtain that

$$\omega_2 = \alpha_1 + .. + \alpha_{l-1} \in \Delta\,(l - 1)$$

and $\omega_1 + \omega_2 = \delta - \alpha_2 \in \Phi$.

3. $\mathfrak{g} = C_l$: Consider the maximal root $\delta = 2\alpha_1 + .. + 2\alpha_{l-1} + \alpha_l$ and the particular root α_1. Now

$$(\delta_1, \alpha_j) = \begin{cases} 1 & \text{for } j = 1 \Rightarrow \omega_1 = \delta_1 - \alpha_1 \in \Delta\,(l - 1) \\ 1 & \text{for } j = l \ \Rightarrow \omega_2 = \delta_1 - \alpha_l \in \Delta\,(l - 1) \end{cases}$$

and $\omega_1 + \omega_2 = \delta - \alpha_1 \in \Delta\,(2l - 2)$.

4. $\mathfrak{g} = D_l$: As before, we have $\delta_1 - \alpha_1, \delta_1 - \alpha_l \in \Phi$. Considering the α_2-string through $\delta_1 - \alpha_1$ and the α_{l-1}-sting through $\delta_1 - \alpha_l$ we obtain $\omega_1 = \delta_1 - \alpha_1 - \alpha_2, \omega_2 = \delta_1 - \alpha_l - \alpha_{l-1} \in \Delta\,(l - 2)$ and $\omega_1 + \omega_2 = \delta - \alpha_2 \in \Delta\,(2l - 4)$.

5. $\mathfrak{g} = E_6$: $ht\,(\delta) = 11$

$$\omega_1 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \in \Delta\,(5)$$
$$\omega_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \in \Delta\,(5)$$

where $\omega_1 + \omega_2 = \delta - \alpha_2$.

6. $\mathfrak{g} = E_7$: $ht\,(\delta) = 17$

$$\omega_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \in \Delta\,(8)$$
$$\omega_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 \in \Delta\,(8)$$

$\omega_1 + \omega_2 = \delta - \alpha_1$.

7. $\mathfrak{g} = E_8$: $ht\,(\delta) = 29$

$$\omega_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \in \Delta\,(14)$$
$$\omega_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 \in \Delta\,(14)$$

$\omega_1 + \omega_2 = \delta - \alpha_8$.

8. $\mathfrak{g} = F_4 : ht(\delta) = 11$

$$\omega_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \in \Delta(5)$$
$$\omega_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4 \in \Delta(5)$$

$\omega_1 + \omega_2 = \delta - \alpha_1.$

$\square$

Remark 7. *We see that the classical theory provides a lot of naturally graded Lie algebras satisfying the centralizer property. However, it is usually unconvenient to manipulate these algebras, because of the great difference between its dimension and nilpotence class : the first is too high in comparison with the last.*

Note. *Unless otherwise stated, whenever we speak in future about Lie algebras $\mathfrak{g}$ satisfyng (P), we will understand that $\mathfrak{g}$ is naturally graded and satisfies condition 2) and not 1).*

2.2 $(P2)$-algebras

With the conventions adopted, it follows immediately from the preceding remarks :

Proposition 2. *A $(P2)$ Lie algebra $\mathfrak{g}$ is filiform if and only if $\mathfrak{g} \simeq Q_n$ for $n \geq 6$.*

Recall that the set of characteristic sequences for a nilpotent Lie algebra $\mathfrak{g}$ is ordered lexicographically. As $(n-1, 1)$ is the maximum of this set, it is natural to begin our study with its immediate successor, i.e, the sequence $(n-2, 1, 1)$, where $n = dim(\mathfrak{g})$. If X_1 is a characteristic vector, then we can find a basis $\{X_1, ..., X_n\}$ such that $[X_1, X_i] = X_{i+1}$, $2 \leq i \leq n-2$. Let ω_1 be the vector of the dual base which corresponds to X_1. It follows from the brackets that the exterior product $\omega_1 \wedge \omega_{i-1}$ is a summand of the differential form $d\omega_i$. If $\mathfrak{g}$ satisfies $(P2)$, then the existence of a vector X_j $(j \geq 2)$ with $[X_{k+1}, X_j] \neq 0$ and $[X_j, X_{j+t}] = 0$, for all $t \geq 1$ suffices.

Proposition 3. *Let $\mathfrak{g}$ be a n-dimensional Lie algebra with characteristic sequence $(n-2, 1, 1)$. If $X_n \in \mathfrak{g}_{2t}$, $1 \leq t \leq \frac{n-2}{2}$ then $\mathfrak{g}$ is not naturally graded.*

Proof. We have, for any t, $\mathfrak{g}_{2t} = \frac{C^{2t-1}\mathfrak{g}}{C^{2t}\mathfrak{g}}$. If $X_n \in \mathfrak{g}_{2t}$ we have the brackets

$$[X_2, X_{2t}] = \lambda_{2t-2}X_{2t+1} + \mu_1 X_n$$
$$[X_j, X_{2t-j+2}] = \lambda^j_{2t-2}X_{2t+1} + \mu^t_1 X_n$$

Applying the adjoint operator $ad(X_1)$ we obtain the condition :

$$[X_1, [X_t, X_{t+1}]] + [X_t, X_{t+2}] = 0$$

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Now $ad(X_t)(\langle X_1,..,X_{t+1}\rangle_{\mathbb{C}}) \cap \langle X_n\rangle = \{0\}$, so that $\mu_1^t = 0$ for all t. On the other hand, the previous condition implies $\mu_1 = (-1)^{t-1}\mu_1^t \ \forall t$, so $X_n \notin C^1\mathfrak{g}$, contradiction with the assumption. $\qquad\blacksquare$

Remark 8. *As a consequence of the previous result, the position of a vector X_n is never optimal in an even indexed graduation block. For this reason, it is convenient to introduce the following convention : for a n-dimensional nilpotent Lie algebra $\mathfrak{g}$ with basis $\{X_1,..,X_n\}$ we say that the vector X_n has depth k, noted $h(X_n) = k$ $\left(1 \leq k \leq \left[\frac{n-3}{2}\right]\right)$, if $X_n \in \mathfrak{g}_{2k+1}$.*
In fact, this definition can be extended to any vector of $\mathfrak{g}$. Observe that there are fractional depths.

Theorem 2. *There do not exist even dimensional (P2) Lie algebras $\mathfrak{g}$ of characteristic sequence $(2m-2,1,1)$.*

Proof. The characteristic sequence imposes the existence of a basis $\{X_1,..,X_n\}$ such that

$$adX_1(X_i) = X_{i+1}, \ 2 \leq i \leq 2m-2$$

thus we have

$$C^k\mathfrak{g} \supset \langle X_{k+2},..,X_{2m-1}\rangle, \ 1 \leq k \leq 2m-3$$

The central descending sequence induces the following relations for the associated graduation

$$\mathfrak{g}_1 \supset \langle X_1, X_2\rangle, \ \mathfrak{g}_k \supset \langle X_{k+1}\rangle, \ 2 \leq k \leq 2m-2$$

If $(P2)$ is satisfied, there exist two nonzero vectors $X,Y \in C^{m-2}\mathfrak{g}$ such that $[X,Y] \neq 0$. Without loss of generality we can suppose $X = X_m$, $Y = X_{m+t}$ for $t \geq 1$. Then

$$[X_m, X_{m+t}] \in [\mathfrak{g}_{m-1}, \mathfrak{g}_{m+t-1}] \subset \mathfrak{g}_{2m-2+t} = \{0\}, \ t \geq 1$$

This shows that the unique admissible graduation block for X_{2m} is $\mathfrak{g}_{m-1}$. Suppose therefore that $X_{2m} \in \mathfrak{g}_{m-1}$. Then $(P2)$ implies $[X_{2m}, X_m] = \lambda X_{2m-1}$ for a nonzero value λ; moreover, m must be even, $m = 2r$. As X_{2m} belongs to the commutator algebra, there exist two indexes $i,j \geq 1$ with $i+j = m-1$ and a pair of vectors $X_{i+1} \in \mathfrak{g}_i$, $X_{j+1} \in \mathfrak{g}_j$ such that $[X_{i+1}, X_{j+1}] = \lambda_{i,j}X_{2m}$, where $\lambda_{i,j}$ is nonzero. Let (i_0, j_0) be the minimal pair with this property; it is not difficult to see that it is $(2, 2r-3)$. Then the associated differential form to the vector X_{2m} is of the following type :

$$d\omega_{2m} = \sum_{t \geq 0} \lambda_{1+t, 2r-4-t}\omega_{2+t} \wedge \omega_{2r-3-t}$$

On the other hand

$$d\omega_{2m-1} = \sum_{s \geq 0} \alpha_t \omega_{2+s} \wedge \omega_{2m-3-t} + \lambda \omega_m \wedge \omega 2m$$

It is immediate to verify the nonexistence of nonzero coefficients $\lambda_{1+t,2r-4-t}$ such that the previous forms satisfy simultaneously

$$d^2\omega_{2m-1} = d^2\omega_{2m} = 0$$

$\square$

Remark 9. *The obstruction for the even dimensional is the same as the one observed in the analysis of filiform algebras. In this sense, the (odd) dimensional Lie algebras which verify (P2) will play the same role as Q_n does for the filiform algebras.*

Now we approach the classification problem : to obtain all nilpotent Lie algebras that satisfy (P2) and whose characteristic sequence is $(2m-1,1,1)$. To avoid trivial cases, the algebras are supposed to be nonsplit.

For $m \geq 4$ let $\mathfrak{s}_m$ be $2m$-dimensional Lie algebra whose structural equations over the basis $\{\omega_1, .., \omega_{2m-1}, \omega_{2m+1}\}$ of $\left(\mathbb{C}^{2m}\right)^*$ are

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 2m-3$$

$$d\omega_{2m-2} = \omega_1 \wedge \omega_{2m-3} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \, \omega_j \wedge \omega_{2m-1-j}$$

$$d\omega_{2m-1} = \omega_1 \wedge \omega_{2m-2} - (m-2)\,\omega_2 \wedge \omega_{2m+1} + \sum_{j=2}^{m-1} (-1)^j \, (m-i)\,\omega_j \wedge \omega_{2m-j}$$

$$d\omega_{2m+1} = \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \, \omega_j \wedge \omega_{2m-1-j}$$

It follows immediately that $\mathfrak{s}_m$ is naturally graded of characteristic sequence $(2m-2,1,1)$ for any $m \geq 4$.

Notation 3. *Consider $\mathbf{E}_{c,1}(\mathfrak{g})$ and let $p \in \{n(\mathfrak{g}), n(\mathfrak{g})+1\}$. Denote by $\mathbf{E}_{c,1}(\mathfrak{g}, p)$ the extensions whose nilindex is p.*

Proposition 4. *Let $m \geq 4$ and $e(\mathfrak{s}_m) \in \mathbf{E}_{c,1}^{2m-1}(\mathfrak{s}_m)$. Then $e(\mathfrak{s}_m)$ satisfies (P2) if and only if it is isomorphic to the Lie algebra $\mathfrak{g}_{(m,m-2)}^3$ given by :*

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 2m-3$$
$$d\omega_{2m-2} = \omega_1 \wedge \omega_{2m-3} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \ \omega_j \wedge \omega_{2m-1-j}$$
$$d\omega_{2m-1} = \omega_1 \wedge \omega_{2m-2} - (m-2)\,\omega_2 \wedge \omega_{2m+1} + \sum_{j=2}^{m-1} (-1)^j\,(m-j)\ \omega_j \wedge \omega_{2m-j}$$
$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} - (m-2)\,\omega_3 \wedge \omega_{2m+1} + \sum_{j=3}^{m} \frac{(-1)^j\,(j-2)\,(2m-1-j)}{2} \ \omega_j \wedge \omega_{2m+1-j}$$
$$d\omega_{2m+1} = \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \ \omega_j \wedge \omega_{2m-1-j}$$

for $m \geq 5$.

If $m = 4$ there is an additional extension $\mathfrak{g}^1_{(4,2)}$:

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 5$$
$$d\omega_6 = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_7 = \omega_1 \wedge \omega_6 + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5 - 2\omega_2 \wedge \omega_9$$
$$d\omega_8 = \omega_1 \wedge \omega_7 + \omega_2 \wedge \omega_7 - \omega_3 \wedge \omega_6 + 2\omega_4 \wedge \omega_5 - 2\omega_3 \wedge \omega_9$$
$$d\omega_9 = \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$

Proof. Let $\{\omega_1,..,\omega_{2m},\omega_{2m+1}\}$ be a basis of $e\,(\mathfrak{s}_m)$ over $\left(\mathbb{C}^{2m+1}\right)^*$ and $\{X_1,..,X_{2m+1}\}$ its dual basis.

Any central extension is specified by the adjunction of a differential form $d\omega_{2m}$. The graduation forces the depth of X_{2m} to be $h\,(X_{2m}) = m-2$. So $d\omega_{2m}$ is of the following type

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m-1} \varphi_{j,2m-j}\,\omega_j \wedge \omega_{2m+1-j} + \varphi_{2,2m+1}\omega_2 \wedge \omega_{2m+1}$$

where $\varphi_{j,2m-j} \in H_2^{2,m-1}\,(\mathfrak{s}_m)$ for $j = 2,..,\left[\frac{2m+1}{2}\right]$ and $\varphi_{2,2m+1} \in H_4^{2,m-1}\,(\mathfrak{s}_m,\mathbb{C})$. The structure of $\mathfrak{s}_m$ implies $0 \neq \varphi_{2,2m+1}$. Moreover, the following relations hold

$$(j-2)\,(2m-1-j)\,\varphi_{3,2m-2} + 2\,(-1)^j\,(m-2)\,\varphi_{j,2m-j} = 0, \ j = 4,..,\left[\frac{2m+1}{2}\right]$$

$$\frac{m}{2}\,(m-3)\,\varphi_{m-1,m+1} + \varphi_{m-1,m+2} = 0$$

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from which we deduce, by the structure of $\mathfrak{s}_m$, that $\varphi_{2,2m-1} = 0$. Observe in particular that the nullity of this cocycle implies the existence of a unique extension. Through an elementary change of basis it follows that this extension is isomorphic to $\mathfrak{g}^3_{(m,m-1)}$ for $m \geq 5$.

An algebra $e_{c,1}(\mathfrak{s}_4) \in \mathbf{E}_{c,1}(\mathfrak{s}_4)$ is determined by the adjunction of a differential form $d\omega_8$. As the nilindex $p = 8$ is fixed, this implies that $h(X_8) = 3$. Then this form must be of the following type :

$$d\omega_8 = \omega_1 \wedge \omega_7 + \varphi_{27}\omega_2 \wedge \omega_7 + \varphi_{36}\omega_3 \wedge \omega_6 + \varphi_{45}\omega_4 \wedge \omega_5 + \varphi_{29}\omega_2 \wedge \omega_9$$

where $\varphi_{27}, \varphi_{36}, \varphi_{45} \in H_2^{2,3}(\mathfrak{s}_4, \mathbb{C})$ and $0 \neq \varphi_{29} \in H_4^{2,3}(\mathfrak{s}_4, \mathbb{C})$. The determinant cocycle is φ_{27} : if it is nonzero we obtain

$$X_2 \wedge X_7 - X_3 \wedge X_6, 2X_2 \wedge X_7 + X_4 \wedge X_5 \in \Omega$$

and otherwise

$$3X_3 \wedge X_6 + 2X_4 \wedge X_5 \in \Omega$$

Thus there are two nonequivalent extensions, the first being isomorphic to $\mathfrak{g}^1_{(4,2)}$ and the second to $\mathfrak{g}^2_{(4,2)}$. $\qquad\square$

Theorem 3. *If $h(X_{2m+1}) = t$, $t \neq m - 2$, any naturally graded Lie algebra $\mathfrak{g}$ with characteristic sequence $(2m - 1, 1, 1)$ that satisfies (P2) is a central extension of either Q_{2m-1} or L_{2m-1}.*

Proof. Let $h(X_{2m+1}) = t$, $t \neq m - 2$. We define the cocycle $\varphi_{j2m+1} \in H^2(\mathfrak{g}, \mathfrak{g})$ by

$$\varphi_{j2m+1}(X_j, X_{2m+1}) = \begin{cases} \alpha_{j2m+1}X_{j+1+2t} & \text{if } j + 2t \leq 2m - 1 \\ 0 & \text{if } j + 2t > 2m - 1 \end{cases}$$

The action of the adjoint operator $ad(X_1)$ implies the conditions

$$\alpha_{22m+1} = -\alpha_{32m+1} = ... = (-1)^{k_0} \alpha_{k_0 2m+1}$$

where k_0 is the last value for which φ_{j2m+1} is nonzero. Moreover,

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{i,j} \varphi_{ij}\,\omega_i \wedge \omega_j + \alpha_{2,2m+1}\omega_2 \wedge \omega_{2m+1}$$

where $\varphi_{ij} \in H_2^{2,m-1}(\mathfrak{g}, \mathbb{C})$. The Jacobi condition $d(d\omega_{2m}) = 0$ implies $\alpha_{2,2m+1} = 0$, so that the cocycle φ_{j2m+1} is identically zero. Thus the vector X_{2m+1} is central and the factor algebra $\frac{\mathfrak{g}}{\langle X_{2m+1} \rangle}$ is naturally graded and filiform, isomorphic to Q_{2m-1} if $t \neq m - 1$ and isomorphic to L_{2m-1} if $t = m - 1$. $\qquad\square$

Remark 10. *We commented the existence of vectors having fractionary depth, according to the definition given before. To cover all cases, it must be shown that for these depths there do not exist extensions which satisfy the required conditions.*

Proposition 5. *For $m \geq 4$ and $q \equiv 0 \, (\mathrm{mod} \, 2)$*

$$\mathbf{E}_{c,1}^{\frac{q+1}{2},2} \left(Q_{2m-1}, 2m - 1 \right) = 0$$

Proof. An extension $e\left(Q_{2m-1} \right) \in \mathbf{E}_{c,1}^{\frac{q+1}{2},2} \left(Q_{2m-1}, p \right)$ is determined by the cocycles of the space $H_2^{2,\frac{q+1}{2}} \left(Q_{2m-1}, \mathbb{C} \right)$. Then the differential form $d\omega_{2m+1}$ is of type

$$d\omega_{2m+1} = \sum_{i,j} \varphi_{ij} \, \omega_i \wedge \omega_j$$

where the indexes i, j satisfy

$$i + j = q + 4$$

As q is even, let $q = 2t$. The the form $d\omega_{2m+1}$ can be rewritten as

$$d\omega_{2m+1} = \sum_{j=2}^{t+1} \varphi_{j,4+2t-j} \, \omega_j \wedge \omega_{4+2t-j}$$

It is trivial to verify that the equations

$$\varphi_{2,2+2t} + (-1)^j \, \varphi_{j,4+2t-j} = 0, \ 3 \leq j \leq t + 1$$

are satisfied. This allows us to take a common factor, so that

$$d\omega_{2m+1} = \varphi_{2,2+2t} d\varpi_t$$

where this form is easily proven to be nonclosed. So we deduce the nonexistence of naturally graded with the required nilindex in $\mathbf{E}_{c,1}^{\frac{q+1}{2},2} \left(Q_{2m-1} \right)$. $\qquad \square$

Proposition 6. *For $m \geq 4$ and $q \equiv 0 \, (\mathrm{mod} \, 2)$*

$$\mathbf{E}_{c,1}^{\frac{q+1}{2},2} \left(L_{2m-1}, 2m - 1 \right) = 0$$

The proof is analogous to the previous case.

Corollary 1. *For $m \geq 5, r \geq 2$ and $k_1, .., k_r \in \mathbb{Z}^+$*

$$\mathbf{E}_{c,1}^{\frac{q+1}{2},k_1,..,k_r} \left(Q_{2m-1}, 2m - 1 \right) = \mathbf{E}_{c,1}^{\frac{q+1}{2},k_1,..,k_r} \left(L_{2m-1}, 2m - 1 \right) = \{0\}$$

Theorem 4 (Classification of $(P2)$-algebras with ch.s. $(2m-1, 1, 1)$). *A naturally graded Lie algebra $\mathfrak{g}$ with characteristic sequence $(2m-1, 1, 1)$ is a $(P2)$-algebra if and only if it is isomorphic to one of the following models :*

1. $\mathfrak{g}^1_{(4,2)}$:

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_j, \ j = 3, 4, 5$$
$$d\omega_6 = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_5 - \omega_2 \wedge \omega_4$$
$$d\omega_7 = \omega_1 \wedge \omega_6 + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5 - 2\omega_2 \wedge \omega_9$$
$$d\omega_8 = \omega_1 \wedge \omega_7 + \omega_2 \wedge \omega_7 + \omega_3 \wedge \omega_6 - 2\omega_4 \wedge \omega_5 - 2\omega_3 \wedge \omega_9$$
$$d\omega_9 = \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$

2. $\mathfrak{g}^2_{(m,t)}$ $(1 \leq t \leq m - 2)$:

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 2m - 1$$
$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \ \omega_j \wedge \omega_{2m+1-j}$$
$$d\omega_{2m+1} = \sum_{j=2}^{t+1} (-1)^j \ \omega_j \wedge \omega_{3-j+2t}$$

3. $\mathfrak{g}^3_{(m,m-2)}$:

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 2m - 3$$
$$d\omega_{2m-2} = \omega_1 \wedge \omega_{2m-3} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \ \omega_j \wedge \omega_{2m-1-j}$$
$$d\omega_{2m-1} = \omega_1 \wedge \omega_{2m-2} + (m-2) \omega_2 \wedge \omega_{2m+1} + \sum_{j=2}^{m-1} (-1)^j (m-j) \ \omega_j \wedge \omega_{2m-j}$$
$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + (m-2) \omega_3 \wedge \omega_{2m+1} + \sum_{j=3}^{m} \frac{(-1)^j (j-2)(2m-1-j)}{2} \ \omega_j \wedge \omega_{2m+1-j}$$
$$d\omega_{2m+1} = \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \ \omega_j \wedge \omega_{2m-1-j}$$

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4. $\mathfrak{g}^4_{(m,m-1)}$:

$$dω_1 = dω_2 = 0$$
$$dω_j = ω_1 ∧ ω_{j-.1},\ 3 ≤ j ≤ 2m$$
$$dω_{2m+1} = \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j\, ω_j ∧ ω_{2m+1-j}$$

Moreover, these algebras are pairwise non isomorphic.

The proof will be a consequence of the next results :

Lemma. *For $m ≥ 4$ the following equations hold*

1. $\mathbf{E}_{c,1}\,(Q_{2m-1}, 2m-1) = \sum_{t=1}^{t-2} \mathbf{E}^{t,2}_{c,1}\,(Q_{2m-1}, 2m-1)$

2. $\mathbf{E}_{c,1}\,(L_{2m-1}, 2m-1) = \mathbf{E}^{m-1,2}_{c,1}\,(L_{2m-1}, 2m-1).$

Proof. It is not difficult to see that if $k ≠ 2$, then

$$\mathbf{E}^{t,k}_{c,1}\,(\mathfrak{g}, 2m-1) = 0,\ \mathfrak{g} = L_n\ \text{or}\ \ Q_n$$

For $k = 2$ and any of the nongiven t's the nonexistence of naturally graded extensions with the required nilindex is routine. The remaining cases are a direct consequence of the previous results. □

Proposition 7. *For $m ≥ 4$ any extension $\mathfrak{g}' ∈ \mathbf{E}^{t,2}_{c,1}\,(Q_{2m-1}, 2m-1)$ is isomorphic to $\mathfrak{g}^2_{m,t}$ if $1 ≤ t ≤ m-2$. For $t < 1$ and $t ≥ m-1$ $\mathbf{E}^{t,2}_{c,1}\,(Q_{2m-1}, 2m-1) = 0$.*

Proof. For $1 ≤ t ≤ m-2$ the cocycles $φ_{ij} ∈ H^{2,t}_2\,(Q_{2m-1}, \mathbb{C})$ must satisfy the relation $i + j = 2t + 3$. It is immediate to verify that this space is generated by the cocycles

$$φ_{2,2t+1}, φ_{3,2t}, ..., φ_{t+1,t+2}$$

subjected to the relations

$$φ_{2,2t+1} + (-1)^{j-1}\, φ_{j,2t+3-j} = 0,\ j = 3, .., t+1$$

If $\{X_1, .., X_{2m}\}$ is the dual base of $\{ω_1, .., ω_{2m}\}$, we have

$$X_{2,2t+1}, X_{3,2t}, ..., X_{t+1,t+2} ∈ Ker\,λ$$

and

$$X_{2,2t+1} + (-1)^{j-1}\, X_{j,2t+3-j} ∈ Ω,\ j = 3, .., t+1$$

Thus there is, for any t, only one extension, which is isomorphic to $\mathfrak{g}^2_{(m,t)}$. For the remaining values of t it is easy to see that Q_{2m-1} does not admit naturally graded extensions with the prescribed characteritic sequence. □

Proposition 8. *For $m \geq 4$ any extension $\mathfrak{g}' \in \mathbf{E}_{c,1}^{t,2}(L_{2m-1}, 2m-1)$ is isomorphic to $\mathfrak{g}_{m,m-1}^4$ if $t = m-1$. For $t \neq m-1$ $\mathbf{E}_{c,1}^{t,2}(L_{2m-1}, 2m-1) = 0$.*

Proof. Similarly to the previous case we have

$$X_j \wedge X_{2m-j} \in Ker\,\lambda, \; j = 2, .., \left[\frac{2m+1}{2}\right]$$

and

$$X_2 \wedge X_{2m-2} + (-1)^{j-1} X_j \wedge X_{2m-j} \in \Omega, \; j = 3, .., \left[\frac{2m+1}{2}\right]$$

so that there exists a unique extension, isomorphic to $\mathfrak{g}_{(m,m-1)}^4$. $\qquad\blacksquare$

Proof of theorem 4. $\implies$) We can suppose $m \geq 5$, as we have studied the case $m = 4$ before. We know that if the depth of the vector X_{2m+1} is $h(X_{2m+1}) = t \in \mathbb{Z}$, $t \neq m-2$ the factor algebra $\frac{\mathfrak{g}}{\langle X_{2m+1}\rangle}$ is naturally graded and filiform, thus $\mathfrak{g}$ is a central extension of either Q_{2m-1} or L_{2m-1}. Let also be $h(X_{2m+1}) = m-2$. If X_{2m+1} is central, we obtain again a central extension of Q_{2m-1}. If not, then the differential form $d\omega_{2m}$ has a nonzero coefficient associated to the summand $\omega_2 \wedge \omega_{2m+1}$. In this case, the central element to be taken is X_{2m}, and it is not difficult to see that $\frac{\mathfrak{g}}{\langle X_{2m}\rangle}$ is a naturally graded Lie algebra isomorphic to $\mathfrak{s}_m$. As the central graded extensions of this algebra which increment the nilindex in one unity are unique, this algebra must be isomorphic to $\mathfrak{g}_{(m,m-2)}^3$.

Finally, for the fractionary depths we have seen the nonexistence of extensions of this type.

$\impliedby$) It is a trivial verification that the models satisfy the requirements. $\qquad\blacksquare$

3 Classification of $(P2)$-algebras with characteristic sequence $(2m-1, 2, 1)$

In this section we use the preceding results to establish a classification of $(P2)$-algebras when the second entry of the characteristic sequence is increased by one. We will see that, with one exception, these algebras are obtained by considering central extensions of the preceding models.

Let $(2m-1, 2, 1)$ be the characteristic sequence of $\mathfrak{g}$ and X_1 a charateristic vector. Then we can find a basis $\{X_1, .., X_{2m+2}\}$ dual to the base $\{\omega_1, .., \omega_{2m+2}\}$ and such that $[X_1, X_i] = X_{i+1}, 2 \leq i \leq 2m-1$ and $[X_1, X_{2m+1}] = X_{2m+2}$.

Remark 11. *In contrast to the preceding case, we will see that now there are split algebras which admit nonsplit naturally graded central extensions of degree one which satisfy $(P2)$. This will be justified by the existence of a two dimensional Jordan block for the adjoint operator for a characteristic vector.*

For $m \geq 4$, let $\mathfrak{g}^{1+k}_{(m,0)}$ $(k = 0, 1)$ be the algebras whose Cartan-Maurer equations over the basis $\{\omega_1, .., \omega_{2m+2}\}$ are :

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 2m - 1$$
$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \, \omega_j \wedge \omega_{2m+1-j}$$
$$d\omega_{2m+1} = 0$$
$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + k\omega_2 \wedge \omega_{2m+1}$$

which are clearly $(m-1)$-abelian.

Lemma. *For $m \geq 4$, a nosnplit naturally extension of $Q_n \oplus \mathbb{C}$ satisfies $(P2)$ if and only if it is isomorphic to either $\mathfrak{g}^{1+k}_{(m,0)}$ for $k = 0, 1$.*

Proof. Bot the graduation and $(P2)$ imply that the only cocycles that must be considered are those belonging to the space $H^{2,1}_{2m-1}(Q_{2m-1} \oplus \mathbb{C})$. Thus the only cohomology classes that give central extensions with the prescribed conditions are $\varphi_{1,2m+1}$ and $\varphi_{2,2m+1}$. The differential form $d\omega_{2m+2}$ associated to the koined vector X_{2m+2} has the form :

$$d\omega_{2m+2} = \alpha\varphi_{1,2m+1}\omega_1 \wedge \omega_{2m+1} + \beta\varphi_{2,2m+1}\omega_2 \wedge \omega_{2m+1}$$

where $\alpha, \beta \in \mathbb{C}$. Clearly α must be nonzero, and by a change of basis we can suppose $\alpha = 1$. If $\beta = 0$ we obtain $\mathfrak{g}^1_{(m,0)}$, while for nonzero β we obtain $\mathfrak{g}^2_{(m,0)}$. $\qquad\blacksquare$

Remark 12. *Observe that with the definition of depth introduced earlier, the vector X_{2m+2} of an algebra $\mathfrak{g}$ of characteristic sequence $(2m-1, 2, 1)$ has fractionary depth. This follows directly from it, as the position of this last vector is determined by the one of X_{2m+1}, as these vectors form the two dimensional Jordan box for $\mathrm{ad}(X_1)$.*

Theorem 5. *Let $m \geq 4$. If $h(X_{2m+2}) = \frac{2t+1}{2}$ $(3 \geq t \geq m - 2)$, then a $(P2)$-algebra of characteristic sequence $(2m-1, 2, 1)$ is an extension of $\mathfrak{g}^2_{(m,t)}$ or $\mathfrak{g}^3_{(m,m-1)}$.*

The proof is of the same kind as theorem 3.
For the lowest values of t, a similar result holds. However, here we find additional extensions or pathological cases, which justify a separated treatment. For $m \geq 4$ consider the Lie algebra $\mathfrak{s}^1_m$ expressed over the basis $\{\omega_1, .., \omega_{2m-1}, \omega_{2m+1}, \omega_{2m+2}\}$:

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1},\ 3 \le j \le 2m-1$$
$$d\omega_{2m+1} = \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5$$

Remark 13. *This algebra plays the same role as $\mathfrak{s}_m$ before. Observe also that its characteristic sequence is $(2m-2, 2, 1)$. This case corresponds to those models for which the "last" vector is not central.*

Proposition 9. *Let $\mathfrak{g}$ be a $(P2)$-algebra of characteristic sequence $(2m-1, 2, 1)$ and $h(X_{2m+2}) = \frac{2t+1}{2}$ with $t = 1, 2$ over the ordered basis $\{X_1, .., X_{2m+2}\}$. Then $\mathfrak{g}$ is a central extension of $\mathfrak{g}^2_{(m,1)}$ if $t = 1$, a central extension of $\mathfrak{g}^1_{(4,2)}$ if $t = 2, m = 4$, or a central extension of $\mathfrak{g}^2_{(m,2)}$ or $\mathfrak{s}_m$ if $t = 2$ and $m \ge 4$.*

Proof. Again, the main idea of the proof is the same as in theorem 3. We only comment few aspects : for the exceptional (nine dimensional) case $\mathfrak{g}^1_{(4,1)}$ a central extension satisfying $(P2)$ is determined by the cocycles $\varphi_{19} \in H_4^{2,\frac{5}{2}}\left(\mathfrak{g}^3_{(4,2)}, \mathbb{C}\right)$, $\varphi_{26}, \varphi_{35} \in H_2^{2,\frac{5}{2}}\left(\mathfrak{g}^3_{(4,2)}, \mathbb{C}\right)$ and $\varphi_{29} \in H_5^{2,\frac{5}{2}}\left(\mathfrak{g}^3_{(4,2)}, \mathbb{C}\right)$ subjected to the relations

$$\varphi_{26} + 2\varphi_{35} = 0$$
$$\varphi_{19} + \varphi_{35} = 0$$
$$2\varphi_{19} + \varphi_{29} = 0$$

It is clear that they define a unique extension.
Any central extension of degree one of $\mathfrak{s}^1_m$ is determined by the adjunction of a differential form, which we will call $d\omega_{2m}$. The graduation and the characteristic sequence imply that this differential form is of the type

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{i,j} \varphi_{ij}\, \omega_i \wedge \omega_j + \varphi_{2,2m+2}\omega_2 \wedge \omega_{2m+2} + \varphi_{3,2m+1}\omega_3 \wedge \omega_{2m+1}$$

where $\varphi_{ij} \in H_2^{2,m-1}\left(G^1_m, \mathbb{C}\right)$, $\varphi_{2,2m+2}, \varphi_{2,2m+1} \in H_5^{2,m-1}\left(G^1_m, \mathbb{C}\right)$, as we have $h(X_{2m}) = m - 1$. The following relations hold

$$\varphi_{2,2m-1} + (-1)^j\, \varphi_{j,2m+1-j} = 0,\ j = 3, .., \left[\frac{2m+1}{2}\right]$$

$$\varphi_{2,2m+2} + \varphi_{3,2m+1} = 0$$

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This implies the existence of a unique extension having characteristic sequence $(2m - 1, 2, 1)$, and given by the equations

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m - 1$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \omega_j \wedge \omega_{2m+1-1} + \omega_2 \wedge \omega_{2m+2} - \omega_3 \wedge \omega_{2m+1}$$

$$d\omega_{2m+1} = \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5$$

We denote this algebra with $\mathfrak{g}^5_{(m,2)}$. $\qquad\qquad\qquad\qquad$ □

Now it is not difficult to establish the main result for this characteristic sequence :

Theorem 6 (Classification of $(P2)$-algebras of ch.s. $(2m - 1, 2, 1)$). *Let $m \ge$ 4 and $\mathfrak{g}$ be a $(2m + 2)$- dimensional Lie algebra of characteristic sequence $(2m - 1, 2, 1)$. Then $\mathfrak{g}$ is a $(P2)$-algebra if and only if it is isomorphic to one of the following algebras :*

1. $\mathfrak{g}^{1,1}_{(4,2)}$:

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 5$$
$$d\omega_6 = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_7 = \omega_1 \wedge \omega_6 + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5 - 2\omega_2 \wedge \omega_9$$
$$d\omega_8 = \omega_1 \wedge \omega_7 + \omega_2 \wedge \omega_7 + \omega_3 \wedge \omega_6 - 2\omega_4 \wedge \omega_5 - 2\omega_3 \wedge \omega_9$$
$$d\omega_9 = \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_{10} = \omega_1 \wedge \omega_9 + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5 - 2\omega_2 \wedge \omega_9$$

2. $\mathfrak{g}^{1+k}_{(m,0)}$ $(k = 0, 1)$:

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m - 1$$
$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \omega_j \wedge \omega_{2m+1-j}$$
$$d\omega_{2m+1} = 0$$
$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + k\omega_2 \wedge \omega_{2m+1}$$

3. $\mathfrak{g}^{2,1}_{(m,t)}$ $(1 \le t \le m-2)$:

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m-1$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \, \omega_j \wedge \omega_{2m+1-j}$$

$$d\omega_{2m+1} = \sum_{j=2}^{t+1} (-1)^j \, \omega_j \wedge \omega_{3-j+2t}$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{t+1} (-1)^j \, (t+2-j) \, \omega_j \wedge \omega_{4-j+2t}$$

4. $\mathfrak{g}^{2,2}_{(m,1)}$:

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m-1$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \, \omega_j \wedge \omega_{2m+1-j}$$

$$d\omega_{2m+1} = \omega_2 \wedge \omega_3$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + \omega_2 \wedge \omega_4 + \omega_2 \wedge \omega_{2m+1}$$

5. $\mathfrak{g}^{3,1}_{(m,m-2)}$:

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m-3$$

$$d\omega_{2m-2} = \omega_1 \wedge \omega_{2m-3} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \, \omega_j \wedge \omega_{2m-1-j}$$

$$d\omega_{2m-1} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m-1} (-1)^j \, (m-j) \, \omega_j \wedge \omega_{2m+1-j} - (m-2)\omega_2 \wedge \omega_{2m+1}$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m} \frac{(-1)^j \, (j-2)\,(2m-1-j)}{2} \, \omega_j \wedge \omega_{2m+1-j}$$

$$d\omega_{2m+1} = \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \, \omega_j \wedge \omega_{2m-1-j}$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{m-1} (-1)^j \, (m-j) \, \omega_j \wedge \omega_{2m-j} - (m-2)\omega_2 \wedge \omega_{2m+1}$$

6. $\mathfrak{g}^5_{(m,2)}$:

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m-1$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \, \omega_j \wedge \omega_{2m+1-j} + \omega_2 \wedge \omega_{2m+2} - \omega_3 \wedge \omega_{2m+1}$$

$$d\omega_{2m+1} = \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5$$

Moreover, these algebras are pairwise non isomorphic.

We proceed stepwise, as done in the previous section.

Proposition 10. *For $m \ge 4$ the following assertions hold :*

1. $\mathbf{E}_{c,1}\left(\mathfrak{g}^2_{(m,1)}, 2m-1\right) = \mathbf{E}^{\frac{3}{2},2,2m-2}_{c,1}\left(\mathfrak{g}^2_{(m,1)}, 2m-1\right) + \mathbf{E}^{\frac{3}{2},2,2m-2,2m-1}_{c,1}\left(\mathfrak{g}^2_{(m,1)}, 2m-1\right)$

2. $\mathbf{E}_{c,1}\left(\mathfrak{g}^2_{(m,t)}, 2m-1\right) = \mathbf{E}^{\frac{2t+1}{2},2,2m-2t}_{c,1}\left(\mathfrak{g}^2_{(m,1)}, 2m-1\right)$

3. $\mathbf{E}_{c,1}\left(\mathfrak{g}^3_{(m,m-2)}, 2m-1\right) = \mathbf{E}^{\frac{2m-3}{2},2,4,5}_{c,1}\left(\mathfrak{g}^3_{(m,m-2)}, 2m-1\right)$

Proof. For any case the reasoning is similar to previous ones.

1. The cocycle $\varphi_{24} \in H^{2,\frac{3}{2}}_2\left(\mathfrak{g}^2_{(m,1)}, \mathbb{C}\right)$ makes reference to the differential form $d\omega_{2m+1} = \omega_2 \wedge \omega_3$. To this we have to add, by the characteristic sequence and the closure of the forms system, the cocycle $\varphi_{1,2m+1} \in H^{2,\frac{3}{2}}_{2m-2}\left(\mathfrak{g}^2_{(m,1)}, \mathbb{C}\right)$, subjected to the condition $\varphi_{1,2m+1} + \varphi_{24} = 0$. A second class of extensions is defined by the cocycle (class) $\varphi_{2,2m+1} \in H^{2,\frac{3}{2}}_{2m-1}\left(\mathfrak{g}^2_{(m,1)}, \mathbb{C}\right)$.

2. The cocycles which define the desired extensions are

$$\varphi_{j,4-j+2t} \in H^{2,\frac{2t+1}{2}}_2\left(\mathfrak{g}^2_{(m,t)}, \mathbb{C}\right), \ \varphi_{1,2m+1} \in H^{2,\frac{2t+1}{2}}_{2m-2t}\left(\mathfrak{g}^2_{(m,t)}, \mathbb{C}\right)$$

satisfying

$$\varphi_{1,2m+1} + \varphi_{t+1,3+t} = 0$$
$$\varphi_{2,2+t} + (-1)^j \, (t+2-j) \, \varphi_{j,4-j+2t} = 0, \ 3 \le j \le t+1$$

3. We have to consider the cocycles

$$\varphi_{1,2m+1} \in H^{2,\frac{2m-3}{2}}_4\left(\mathfrak{g}^3_{(m,m-1)}, \mathbb{C}\right), \ \varphi_{j,2m-j} \in H^{2,\frac{2m-3}{2}}_2\left(\mathfrak{g}^3_{(m,m-1)}, \mathbb{C}\right)$$
$$\varphi_{2,2m+1} \in H^{2,\frac{2m-3}{2}}_5\left(\mathfrak{g}^3_{(m,m-1)}, \mathbb{C}\right)$$

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subjected to the relations

$$(m-2)\,\varphi_{1,2m+1} + \varphi_{2,2m+1} = 0$$
$$\varphi_{1,2m+1} + (-1)^m\,\varphi_{m-1,m+1} = 0$$
$$\varphi_{2,2m-2} + (-1)^j\,(m-j)\,\varphi_{j,2m-j} = 0,\ 3 \le j \le m$$

$\square$

Remark 14. *It follows that for other choices of the superindixes, and in particular for fractionary depths, the previous spaces reduce to zero.*

Corollary 2. *For $m \ge 4$ and $1 \le t \le m-2$ we have*

$$\mathbf{E}_{c,1}^{t,k_1,...,k_r}\left(\mathfrak{g}, 2m-1\right) = \{0\}$$

where $\mathfrak{g} \in \left\{\mathfrak{g}_{(4,2)}^1, \mathfrak{g}_{(m,m-1)}^3, \mathfrak{g}_{(m,t)}^2\right\}$.

Corollary 3. *The following identities hold*

1. *Any extension $\mathfrak{e} \in \mathbf{E}_{c,1}^{\frac{2m-3}{2},2,4,5}\left(\mathfrak{g}_{(m,m-2)}^3, 2m-1\right)$ is isomorphic to $\mathfrak{g}_{(m,m-2)}^{3,1}$.*

2. *Any extension $\mathfrak{e} \in \mathbf{E}_{c,1}^{\frac{3}{2},2,2m-2,2m-1}\left(\mathfrak{g}_{(m,1)}^2, 2m-1\right)$ is isomorphic to $\mathfrak{g}_{(m,1)}^{2,1}$*

3. *Any extension $\mathfrak{e} \in \mathbf{E}_{c,1}^{\frac{2t+1}{2},2,2m-2t}\left(\mathfrak{g}_{(m,t)}^2, 2m-1\right)$ is isomorphic to $\mathfrak{g}_{(m,t)}^{2,1}$*

The proof is elementary.

Now we prove the classification theorem :

Proof of theorem 6 . If $h\left(X_{2m+2}\right) = \frac{1}{2}$, it is trivial to verify that this vector must be in the center. Then the factor algebra $\frac{\mathfrak{g}}{\langle X_{2m+2}\rangle}$ has characteristic sequence $(2m-1,1,1)$ and $h\left(X_{2m+1}\right) = 0$. We know that for this depth there does not exist any nonsplit model. Thus $\frac{\mathfrak{g}}{\langle X_{2m+2}\rangle} \simeq Q_{2m-1} \oplus \mathbb{C}$ and $\mathfrak{g}$ must be isomorphic to either $\mathfrak{g}_{(m,0)}^1$ or $\mathfrak{g}_{(m,0)}^2$. If $h\left(X_{2m+2}\right) = \frac{3}{2}$ the characteristic sequence and the graduation imply that $X_{2m+2} \in Z\left(\mathfrak{g}\right)$, thus $\frac{\mathfrak{g}}{\langle X_{2m+2}\rangle} \simeq \mathfrak{g}_{(m,1)}^2$. In consequence $\mathfrak{g} \in E_{c,1}\left(\mathfrak{g}_{(m,1)}^2\right) \cap \delta N_{2m}^{2m+2}$.

If $h\left(X_{2m+2}\right) = \frac{5}{2}$ the Jacobi conditions give two solutions : if $X_{2m+2} \in Z\left(\mathfrak{g}\right)$ then $\frac{\mathfrak{g}}{\langle X_{2m+2}\rangle}$ is isomorphic to $\mathfrak{g}_{(m,2)}^2$, and if $X_{2m+2} \notin Z\left(\mathfrak{g}\right)$ then X_{2m} must be a central vector, from which $\frac{\mathfrak{g}}{\langle X_{2m}\rangle}$ is isomorphic to G_m^1; in the first case $\mathfrak{g} \simeq \mathfrak{g}_{(m,2)}^{2,1}$ and in the second $\mathfrak{g} \simeq \mathfrak{g}_{(m,2)}^5$.

For $h\left(X_{2m+2}\right) = \frac{2t+1}{2}$, $3 \le t \le m-3$ the characteristic sequence and the graduation imply that $Z\left(\mathfrak{g}\right) \supset \langle X_{2m+2}\rangle$, thus $\mathfrak{g} \simeq \mathfrak{g}_{(m,t)}^2$ by the previous reasoning. Finally, for the depth $\frac{2m-3}{2}$ the factor of $\mathfrak{g}$ by the central ideal $\langle X_{2m}\rangle$ is either isomorphic to $\mathfrak{g}_{(m,m-2)}^2$ or $\mathfrak{g}_{(m,m-2)}^3$. $\square$

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The next table resumes the families obtained in theorem 6 :

Table 1			
$\mathfrak{g}$	$\dim \mathfrak{g}$	ch.s	type
$\mathfrak{g}^1_{(m,0)}$	$2m+2$	$(2m-1,2,1)$	$(3,2,1,..,1)$
$\mathfrak{g}^2_{(m,0)}$	$2m+2$	$(2m-1,2,1)$	$(3,2,1,..,1)$
$\mathfrak{g}^{2,1}_{(m,t)}$	$2m+2$	$(2m-1,2,1)$	$\left(2,1,..,\overset{(2t+1)}{2},2,1,..1\right)$
$\mathfrak{g}^{2,2}_{(m,1)}$	$2m+2$	$(2m-1,2,1)$	$(2,1,2,2,1,..,1)$
$\mathfrak{g}^{3,1}_{(m,m-2)}$	$2m+2$	$(2m-1,2,1)$	$(2,1,..,2,2,1)$
$\mathfrak{g}^5_{(m,2)}$	$2m+2$	$(2m-1,2,1)$	$(2,1,1,1,2,2,1,..,1)$

4 $(P2)$-algebras of characteristic sequence $(2m-1,q,1)$

In this section we describe different families of $(P2)$ Lie algebras in arbitrary dimension and characteristic sequence $(2m-1,q,1)$ with $q \geq 1$.The algebras we enumerate are obtained by central extensions of the algebras classified in theorem 6. Now observe that for any $q \geq 3$ the classification of $(P2)$-algebras having the specified characteristic sequence is given up to the exceptional model (like $\mathfrak{s}_m$ and $\mathfrak{s}^1_m$ before) which appears for any q. The remarkable fact is, however, that for any $q \geq -1$ (here allowing the cases treated) most models can be interpreted as central extensions of the algebra Q_n. This justifies the importance of this model within the $(P2)$-algebras.

Let $m \geq 4$. Consider the Lie algebras

- $\mathfrak{g}^{1+k,q}_{(m,0)}$ $(k=0,1)$, $1 \leq q \leq 2m-3$

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 2m-1$$
$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \, \omega_j \wedge \omega_{2m+1-j}$$
$$d\omega_{2m+1} = 0$$
$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + k\omega_2 \wedge \omega_{2m+1}$$
$$d\omega_{2m+2+r} = \omega_1 \wedge \omega_{2m+1+r} + k\omega_{2+r} \wedge \omega_{2m+1}, \ 1 \leq r \leq q$$

over the basis $\{\omega_1,..,\omega_{2m+2+r}\}$.

- $\mathfrak{g}_{(m,t)}^{2,1,q}$ $(1 \le q \le 2m - 2t - 3)$

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m - 1$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \ \omega_j \wedge \omega_{2m+1-j}$$

$$d\omega_{2m+1} = \sum_{j=2}^{t+1} (-1)^j \ \omega_j \wedge \omega_{3-j+2t}$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{t+1} (-1)^j \, (t + 2 - j) \ \omega_j \wedge \omega_{4-j+2t}$$

$$d\omega_{2m+2+r} = \omega_1 \wedge \omega_{2m+1+r} + \sum_{j=2}^{t+1} (-1)^j \ S_j^r \ \omega_j \wedge \omega_{4-j+2t+r}, \ 1 \le r \le q$$

where

$$S_j^1 = \sum_{k=j}^{t+1} (t + 2 - k), \ 2 \le j \le t + 1$$

$$S_j^k = \sum_{k=j}^{t+1} S_j^{k-1}, \ \ 2 \le k \le q$$

over the basis $\{\omega_1, .., \omega_{2m+2+r}\}$.

- $\mathfrak{g}_{(m,1)}^{2,2,q}$ $(1 \le q \le 2m - 3)$

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m - 1$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \ \omega_j \wedge \omega_{2m+1-j}$$

$$d\omega_{2m+1} = \omega_2 \wedge \omega_3$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + \omega_2 \wedge \omega_4 + \omega_2 \wedge \omega_{2m+1}$$

$$d\omega_{2m+2+r} = \omega_1 \wedge \omega_{2m+1+r} + \omega_2 \wedge \omega_{4+r} + \omega_2 \wedge \omega_{2m+1+r}, \ 1 \le r \le q$$

over the basis $\{\omega_1, .., \omega_{2m+2+r}\}$.

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- $\mathfrak{g}_{(m,2)}^{5,q}\ (1 \le q \le 2m-5)$

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1},\ 3 \le j \le 2m-1$$
$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j\ \omega_j \wedge \omega_{2m+1-j} + \omega_2 \wedge \omega_{2m+1} - \omega_3 \wedge \omega_{2m+1}$$
$$d\omega_{2m+1} = \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + 2\,\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5$$
$$d\omega_{2m+2+r} = \omega_1 \wedge \omega_{2m+1+r} + (2+r)\,\omega_2 \wedge \omega_{6+r} - \omega_3 \wedge \omega_{5+r},\ 1 \le r \le q$$

over the basis $\{\omega_1, .., \omega_{2m+2+r}\}$.

Notation 4. *For $m \ge 4$ and any fixed $q \ge 1$ let $\mathfrak{g}_q \in \left\{ \mathfrak{g}_{(m,0)}^{1+k,q}, \mathfrak{g}_{(m,t)}^{2,1,q}, \mathfrak{g}_{(m,1)}^{2,2,q}, \mathfrak{g}_{(m,2)}^{5,q} \right\}$.*

Theorem 7. *For $q \ge 1$ the Lie algebra $\mathfrak{g}_q$ is a central extension of $\mathfrak{g}_{q-1}$ by $\mathbb{C}$. Moreover, $\mathfrak{g}_q$ is a $(P2)$-algebra of characteristic sequence $(2m-1, 2+q, 1)$.*

Proof. We prove the assertion for $\mathfrak{g}_{(m,t)}^{2,1,q}$. For the remaining cases the reasoning is similar.
Recall that for $\mathfrak{g}_{(m,t)}^{2,1}$ the last differential form is given by

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{t+1} (-1)^j\,(t+2-j)\,\omega_j \wedge \omega_{4-j+2t}$$

A central extension of $\mathfrak{g}_{(m,t)}^{2,1}$ by $\mathbb{C}$ which is a $(P2)$-algebra will be determined by the adjunction of a differential form $d\omega_{2m+3}$, whose structure is

$$d\omega_{2m+3} = \omega_1 \wedge \omega_{2m+2} + \sum_{j=2}^{t+1} (-1)^j\,\varphi_{j,5-j+2t}\,\omega_j \wedge \omega_{5-j+2t},$$

where the cocycles

$$\varphi_{j,5-j+2t} \in H_2^{2,t+1}\left(\mathfrak{g}_{(m,t)}^{2,1}, \mathbb{C}\right)$$

satisfy

$$\varphi_{2,3+2t} + (-1)^j \sum_{k=j}^{t+1} (t+2-k)\,\varphi_{j,5-j+2t} = 0,\ 3 \le j \le t+1$$

We thus obtain a unique extension class which is isomorphic to $\mathfrak{g}^{2,1,1}_{(m,t)}$. This shows the assertion for $q = 1$. Let it be true for $q_0 > 1$. Then the Cartan-Maurer equations of $\mathfrak{g}^{2,1,q_0}_{(m,t)}$ are

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \le j \le 2m - 1$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j \ \omega_j \wedge \omega_{2m+1-j}$$

$$d\omega_{2m+1} = \sum_{j=2}^{t+1} (-1)^j \ \omega_j \wedge \omega_{3-j+2t}$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{t+1} (-1)^j \ (t+2-j) \ \omega_j \wedge \omega_{4-j+2t}$$

$$d\omega_{2m+2+r} = \omega_1 \wedge \omega_{2m+1+r} + \sum_{j=2}^{t+1} (-1)^j \ S_j^r \ \omega_j \wedge \omega_{4-j+2t+r}, \ 1 \le r \le q_0$$

where

$$S_j^1 = \sum_{k=j}^{t+1} (t+2-k), \ 2 \le j \le t+1$$

$$S_j^k = \sum_{k=j}^{t+1} S_j^{k-1}, \ \ 2 \le k \le q$$

Now we extend this algebra by $\mathbb{C}$. Supposing that the extension satisfies the centralizer property and is naturally graded of the prescribed characteristic sequence, the determining cocycles are

$$\varphi_{j,4-j+2t+q_0+1} \quad \in \quad H_2^{2,\frac{2t+2+r}{2}} \left(\mathfrak{g}^{2,1,q_0}_{(m,t)}, \mathbb{C} \right) \text{ if } r \equiv 1 \, (mod \ 2)$$

$$\varphi_{j,4-j+2t+q_0+1} \quad \in \quad H_2^{2,t+\frac{r}{2}+1} \left(\mathfrak{g}^{2,1,q_0}_{(m,t)}, \mathbb{C} \right) \text{ if } r \equiv 0 \, (mod \ 2)$$

We have the relations

$$\varphi_{j,4-j+2t+q_0+1} + (-1)^j \sum_{j=2}^{t+1} S_j^{q_0} \omega_j \wedge \omega_{5-j+2t+q_0} = 0, \ 3 \le j \le t+1$$

and by an elementary change of basis, the adjoined differential form $d\omega_{2m+3+q_0}$ is of type

$$d\omega_{2m+3+q_0} = \omega_1 \wedge \omega_{2m+2+q_0} + \sum_{j=2}^{t+1} (-1)^j S_j^{q_0} \omega_j \wedge \omega_{5-j+2t+q_0}$$

Both the characteristic sequence and centralizer property are obviously satisfied. $\qquad \square$

The algebras $\mathfrak{g}^{3,1}_{(m,m-2)}$ only admit one more extension which is a $(P2)$-algebra. This is due to the extremal position of the vectors that give the two dimensional Jordan block of the characteristic sequence.

Proposition 11. *For $m \geq 4$ the algebra $\mathfrak{g}^{3,1,1}_{(m,m-2)}$ given by*

$$d\omega_1 = d\omega_2 = 0$$

$$d\omega_j = \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 2m - 3$$

$$d\omega_{2m-2} = \omega_1 \wedge \omega_{2m-3} + \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \ \omega_j \wedge \omega_{2m-1-j}$$

$$d\omega_{2m-1} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m-1} (-1)^j (m-j) \ \omega_j \wedge \omega_{2m+1-j} - (m-2) \omega_2 \wedge \omega_{2m+1}$$

$$d\omega_{2m} = \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m} \frac{(-1)^j (j-2)(2m-1-j)}{2} \omega_j \wedge \omega_{2m+1-j} - (m-2) \omega_3 \wedge \omega_{2m+1}$$

$$d\omega_{2m+1} = \sum_{j=2}^{\left[\frac{2m+1}{2}\right]-1} (-1)^j \ \omega_j \wedge \omega_{2m-1-j}$$

$$d\omega_{2m+2} = \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{m-1} (-1)^j (m-j) \ \omega_j \wedge \omega_{2m-j} - (m-2) \omega_2 \wedge \omega_{2m+1}$$

$$d\omega_{2m+3} = \omega_1 \wedge \omega_{2m+2} + \sum_{j=2}^{m-1} (-1)^j S^j \omega_j \wedge \omega_{2m+1-j} - (m-2) \omega_3 \wedge \omega_{2m+1}$$

over the basis $\{\omega_1, .., \omega_{2m+3}\}$ is a $(P2)$-algebra of characteristic sequence $(2m - 1, 3, 1)$.

Proof. Any extensions which satisfies the centralizer property, preserves the graduation and has characteristic sequence $(2m - 1, 3, 1)$ is determined by

$$\varphi_{1,2m+2} \in H_4^{2,m-1}\left(\mathfrak{g}^{3,1}_{(m,m-1)}, \mathbb{C}\right), \ \varphi_{j,2m+1-j} \in H_2^{2,m-1}\left(\mathfrak{g}^{3,1}_{(m,m-1)}, \mathbb{C}\right)$$

$$\varphi_{3,2m+1} \in H_5^{2,m-1}\left(\mathfrak{g}^{3,1}_{(m,m-1)}, \mathbb{C}\right)$$

subjected to the relations

$$(m-2)\varphi_{1,2m+2} + \varphi_{3,2m+1} = 0$$

$$\varphi_{1,2m+2} + (-1)^m \varphi_{m-1,m+2} = 0$$

$$\varphi_{2,2m-2} + (-1)^j S^j \varphi_{j,2m+1-j} = 0, \ 3 \leq j \leq m$$

where $S^j = \sum_{j=2}^{m-1}(m-j)$.
Then the class is unique, and by an elementary change of basis the extended algebra is easily seen to be isomorphic to $\mathfrak{g}^{3,1,1}_{(m,m-2)}$. The centralizer property is given by the form $d\omega_{2m}$. $\square$

We resume the result in the following table :

		Table 2	
$\mathfrak{g}$	$\dim \mathfrak{g}$	ch.s	type
$\mathfrak{g}_{(m,0)}^{1,q}$	$2m+2+q$	$(2m-1,2+q,1)$	$\left(3,2,..,\overset{(2+q)}{2},1,..,1\right)$
$\mathfrak{g}_{(m,0)}^{2,q}$	$2m+2+q$	$(2m-1,2+q,1)$	$\left(3,2,..,\overset{(2+q)}{2},1,..,1\right)$
$\mathfrak{g}_{(m,t)}^{2,1,q}$	$2m+2+q$	$(2m-1,2+q,1)$	$\left(2,1,..,\overset{(2t+1)}{2},2,..,\overset{(q+2t+1)}{2},1,..,1\right)$
$\mathfrak{g}_{(m,1)}^{2,2,q}$	$2m+2+q$	$(2m-1,2+q,1)$	$\left(2,1,2,..,\overset{(3+q)}{2},1,..,1\right)$
$\mathfrak{g}_{(m,2)}^{5,q}$	$2m+2+q$	$(2m-1,2+q,1)$	$\left(2,1,1,1,2,..,\overset{(5+q)}{2},1,..,1\right)$
$\mathfrak{g}_{(m,m-2)}^{3,1,1}$	$2m+3$	$(2m-1,3,1)$	$(2,1,..,1,2,2,2)$

Remark 15. *Finally, the pathological case* $\mathfrak{g}_{(4,2)}^{1,1}$ *admits the extension* $\mathfrak{g}_{(4,2)}^{1,1,1}$ *given by*

$$d\omega_1 = d\omega_2 = 0$$
$$d\omega_j = \omega_1 \wedge \omega_{j-1}; \ 3 \le j \le 5$$
$$d\omega_6 = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_7 = \omega_1 \wedge \omega_6 + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5 - 2\omega_2 \wedge \omega_9$$
$$d\omega_8 = \omega_1 \wedge \omega_7 + \omega_2 \wedge \omega_7 - \omega_3 \wedge \omega_6 + 2\omega_4 \wedge \omega_5 - 2\omega_3 \wedge \omega_9$$
$$d\omega_9 = \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_{10} = \omega_1 \wedge \omega_9 + 2\omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5 - 2\omega_2 \wedge \omega_9$$
$$d\omega_{11} = \omega_1 \wedge \omega_{10} + 3\omega_2 \wedge \omega_7 - \omega_3 \wedge \omega_6 - 2\omega_3 \wedge \omega_9$$

As both the dimension and the characteristic sequence $[(7,3,1)]$ *are fixed, this algebra is not of great interest for the general case.*

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